DET KGL. DANSKE VIDENSKABERNES SELSKAB
MATEMATISK-FYSISKE MEDDELELSER, BIND XXIV, NR. 19

ATOMIC INTERACTION IN PENETRATION PHENOMENA

BY

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KØBENHAVN
I KOMMISSION HOS EJNAR MUNKSGAARD
1948

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Printed in Denmark
Bianco Lunos Bogtrykkeri

§ 1. Introduction.

The theory of the penetration of fast charged particles through matter was originally developed on the basis of an analysis of collisions between the moving particle and individual atoms[1]. In such a treatment, the influence of phase relations between the effects produced in the atoms along the path and, in particular, the resulting coupling between the different encounters is disregarded. This approximation will often be justified, but phenomena do exist for which such atomic interaction effects are of essential importance.

Thus, it was observed by ČERENKOV (1934) that very fast electrons, when passing through dense materials, give rise to a peculiar radiation the properties of which reveal that one has to do, not with independent emission processes by individual excited atoms, but with a radiation emitted coherently by larger portions of the substance. A theory of this phenomenon was developed by FRANK and TAMM (1937) and by TAMM (1939), who showed its immediate connection with the fact that the velocity of the electrons may exceed the rate of propagation of electromagnetic waves in the surrounding medium. Just on account of the interplay of the atoms, the phase velocity of such waves may indeed, over certain spectral regions, be smaller than the velocity of light in vacuo.

The problem of a possible influence of atomic interaction on the stopping and ionization of fast particles was raised by SWANN (1938), who pointed out that the polarization induced by the particle in the matter through which it passes will give rise to a certain screening effect which might, under circumstances,

1) A comprehensive treatment of this subject has recently been given by N. BOHR (1948) and, in the following, we shall often refer to this survey for fuller information regarding general aspects of penetration theory.

reduce the rate of energy loss. The question was treated in detail by FERMI (1939, 1940), who found that the phenomenon may be of great significance for very fast particles, but is in general negligible for non-relativistic velocities. Still, there are exceptions to this rule and, in particular, it has been shown by KRAMERS (1947) that the stopping power of metals is always essentially influenced by the polarization effect.

In the above-mentioned treatments, the matter penetrated is described, in a macroscopic way, as a continuum and since, moreover, on account of the dispersive properties of the medium, which are essential for the phenomena, the electromagnetic fields are resolved in harmonic components, the connection with the simple theory of atomic collisions is somewhat obscure. For this reason, we shall in the following attempt a treatment of the coupling between the encounters from a microscopic point of view so as to bring out as clearly as possible the relationship to ordinary penetration theory. Such an approach is also found to be well suited to obtain simple generalizations of FERMI's formulae.

As a preliminary, we shall in § 2 briefly discuss the arguments which justify a distinction between close and distant collisions of the particle with the atoms in the substance. The mutual interaction of the atoms is of significance only in the distant encounters, the treatment of which is especially simple, since the description may be based entirely on classical mechanical pictures. A few main principles from ordinary stopping theory, of use in the following, are reviewed in § 3, in particular with respect to the influence of the atomic binding forces which effectively limit the radius of action of the penetrating particle by giving the collisions beyond a certain distance an adiabatic character.

In § 4, it is shown how the atomic interaction, from the microscopic point of view, may be regarded as a further screening factor and, especially in the non-relativistic case, may be treated in close analogy to the effect of the atomic binding forces. Moreover, a treatment of the energy loss of the particle is given, in which the stopping power is described as a force with which the atoms in the medium act on the moving particle. The special case of materials containing free electrons is considered more

closely in § 5, where also the influence of damping effects is taken into account. For particle velocities close to that of light, the atomic interaction presents new aspects connected with the retardation of the forces. This problem is discussed in outline in § 6 with reference to corresponding modifications in the relativistic two-body problem considered in Appendix I, and a simple interpretation of the main results of FERMI is obtained.

The Čerenkov radiation and its relation to the stopping power is dealt with in § 7. From the microscopic point of view, the phenomenon simply implies that part of the energy which is intermediately transferred from the particle to the atomic electrons is subsequently emitted as coherent electromagnetic waves. In a macroscopic description, the energy loss of the particle falls naturally into two parts, the first of which is absorbed by the substance in the neighbourhood of the path, while the second is directly transmitted to larger distances in the form of radiation. The distinction leads to some general relations regarding the stopping power for relativistic and non-relativistic particle velocities. In this connection, it is shown in Appendix II how the stopping problem may be treated by the formalism of radiation theory well known from quantum electrodynamics.

While, in the first part of the paper, the atoms are treated as simple dispersion oscillators of a single frequency, more general atomic models are considered in § 8. In this case, exact calculations are rather complicated, but it is shown how the analysis in § 6 makes it possible to obtain a general survey of the phenomena and leads to simple approximate formulae. In § 9, an attempt is made to deduce comprehensive expressions for the influence of atomic interaction on the stopping power of heavy substances. Finally, some experimental data are discussed in § 10. Among the more important applications of the theory is the stopping power of metals and the stopping and ionization of very fast particles; there appears to be satisfactory agreement with the available empirical evidence.

§ 2. Separation between Close and Distant Collisions.

The collisions between a charged particle and an atom must, of course, in general be treated in configuration space by means of proper quantum-mechanical methods, but for a large variety of problems it is permissible to use a simplified procedure in which the particle is described as a centre of force moving along a well-defined path. This description was proved by MOTT (1931) to lead to the same atomic excitation probabilities as the more general treatment, in case of particles of mass large compared with that of the electron and energy great in comparison with the atomic binding energies. Actually, it is a sufficient condition for the adequacy of the procedure that the momentum of the incident particle be large compared with the momentum changes involved in the collision. In fact, the effect of the collision on the atom will, under such circumstances, be approximately independent of the inertial properties of the particle which, thus, acts as if it were infinitely heavy. The condition in question, which is essentially equivalent to the requirement that the wave-length of the particle be small compared with atomic dimensions, is also, from the possibility of representing the particle by a wave-packet, immediately seen to ensure the validity of the simple method.

As regards the problem of the mutual influence of the atoms in the stopping substance, the more violent collisions involving large momentum transfers may obviously be neglected and, in the present connection, we may thus treat the penetrating particle as moving along a fixed path. The atoms may then be specified by their distance from the path, the so-called impact parameter p; as we shall see later in this paragraph, the atomic interaction phenomena will be of importance only for $p \gg a_0$, where a_0 denotes the "radius" of the hydrogen atom, which is a suitable measure for atomic dimensions.

In such distant collisions, the effect of the impact on the atom will, in general[1], amount to only a small perturbation and. since

1) For very large values of the charge of the incident particle combined with a relatively small velocity, special considerations are necessary since then the condition $p \gg a_0$ may not be sufficient to justify a perturbation procedure. Although, in this case, one may proceed by much the same methods (cf. N. BOHR 1948, p. 84), we shall disregard such problems in the present connection, since they may be shown to be quite insignificant under circumstances where the atomic interaction phenomena are of importance.

moreover, for $p \gg a_0$, the perturbing field is approximately constant over atomic dimensions, the encounter may be treated by means of simple mechanical considerations. In fact, as is well known, e. g., from dispersion theory, the atom will behave, with respect to average energy absorption and with respect to the electromagnetic field it generates, like an ensemble of classical harmonic oscillators corresponding to the various excitation and ionization possibilities.

In order to justify that close collisions are disregarded, we have still to make an estimate of the order of magnitude of the distances at which the atomic interaction effects become significant. This interaction arises from the displacement of the atomic electrons during the collision, which turns the neutral atoms into dipoles, and it is evident that the ensuing force K acting on an atomic electron will be comparable with the dipole moment per unit volume, multiplied by the electronic charge $-e$. If $z_1 e$ and v denote the charge and velocity of the particle, and μ the electron mass, the displacement of the electrons during the collision will, for free electrons, be of the order of $z_1 e^2/\mu v^2$ (cf. (3.3)) and tend to be smaller if account is taken of the binding forces. We thus have

$$K \lesssim \frac{z_1 e^2}{\mu v^2} n e^2, \tag{2.1}$$

where n denotes the number of electrons per unit volume. Now, even in dense materials, n never exceeds a_0^{-3} and, introducing $\frac{e^2}{a_0} = \mu v_0^2$, where v_0 is the "velocity" of the electron in the hydrogen atom, it follows that K will be small compared with the direct force of the particle, $\frac{z_1 e^2}{p^2}$, at any rate if

$$p < a_0 \frac{v}{v_0}. \tag{2.2}$$

Thus, for v large compared with v_0, representing the order of magnitude of the "orbital velocity" of the most loosely bound electrons in atoms, it is seen that the polarization effect is of importance only for $p \gg a_0$. In this case we may, therefore, corresponding to the above argumentation, divide the collisions into

two groups with $p < q$ and $p > q$, respectively, where q is chosen in such a manner that, for the first group, the atomic interaction effect is negligible and ordinary penetration theory applies while, for the second group, we are dealing only with distant collisions, which can be treated by classical mechanical methods.

For particle velocities comparable with or smaller than v_0, the penetration phenomena change essentially in character (cf. N. Bohr 1948). In such problems, one need in first approximation consider only the influence of the particle on those atoms through which it actually passes, and the interaction effects will only constitute a minor correction, the taking into account of which would even be a rather spurious refinement due to the difficulties of an accurate treatment of the penetration problem for very slow particles. Throughout the following, we shall therefore confine ourselves to the case of $v \gg v_0$.

§ 3. Collisions between Particle and Single Atom.

Before turning to the problem of the mutual influence of the atoms in penetration phenomena, it will be convenient to review briefly some of the main aspects of a collision between a fast particle and an isolated atom. If, in the first instance, the atomic binding forces are disregarded, we have a pure two-body problem which, in case of distant collisions, is further simplified by the fact that the displacement of the electron during the actual encounter will be small compared with the impact parameter. Moreover, in such encounters, the momentum transfer is always small compared with μc, where c is the light velocity, and we may, therefore, neglect relativity effects as regards the electronic motion.

From symmetry reasons, it follows that the final velocity of the electron will be practically perpendicular to the path of the particle, and for the motion of the electron in this direction we thus have, by means of the well-known expression for the electric field surrounding a uniformly moving point charge,

$$\mu \ddot{\eta} = \frac{z_1 e^2 p \gamma}{(p^2 + \gamma^2 v^2 t^2)^{3/2}}, \tag{3.1}$$

where η is the displacement and where $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$. By simple integrations, (3.1) gives

$$\dot{\eta} = \frac{z_1 e^2}{\mu p v}\left(1 + \frac{\gamma v t}{\sqrt{p^2 + \gamma^2 v^2 t^2}}\right) \tag{3.2}$$

and

$$\eta = \frac{z_1 e^2}{\gamma \mu v^2}\left(\sqrt{1 + \frac{\gamma^2 v^2 t^2}{p^2}} + \frac{\gamma v t}{p}\right). \tag{3.3}$$

As is seen from these expressions, the encounter may be approximately characterized by an effective "collision time" of the order of $p/\gamma v$, during which the acting force is comparable with $z_1 e^2 \gamma / p^2$.

From (3.2) we get in particular

$$T = \frac{1}{2}\mu(\dot{\eta})^2_{t\to\infty} = 2\frac{z_1^2 e^4}{\mu v^2}\frac{1}{p^2} \tag{3.4}$$

for the energy transferred to an electron in a free collision. Since the stopping power of a substance is proportional to the integral of $Tp\,dp$, expression (3.4) cannot, however, be applied for arbitrarily large values of p, and it is thus essential in penetration theory to take into account the factors which tend to restrain the electrons from moving freely. These factors, acting as a kind of screening, may be said to determine a "radius of action" of the particle, representing an upper limit $p_{\max}$, below which the simple expression (3.4) applies. At larger distances, the collisions acquire an increasingly adiabatic character due to the influence of the screening, and the energy transfer will be small compared to that of free encounters. For the energy loss of the particle per unit path, originating in collisions with atoms for which $p > q$, we have accordingly

$$S_q = n\int_q^{p_{\max}} T \cdot 2\pi p\,dp = B \log\frac{p_{\max}}{q}, \tag{3.5}$$

a relation which may be taken to define an effective value of $p_{\max}$. The abbreviation B is given by

$$B = 4\pi\frac{z_1^2 e^4}{\mu v^2} n. \tag{3.6}$$

In the following, a main problem will just be to examine the various screening factors and estimate the corresponding limits of free energy transfer.

In collisions between the particle and isolated atoms, the only screening effect arises from the influence of the atomic binding forces. As mentioned in § 2, we may account for the binding by treating the atom as an ensemble of oscillators of frequencies corresponding to the various transition possibilities. For simplicity, however, we shall in the first instance consider all oscillators to have the same cyclic frequency ω_a; in § 8, we shall return to the problem of more general atomic models.

Besides the force of the particle, there will thus be a binding force of magnitude $\mu\omega_a^2\eta$, acting on the electrons. Of course, the latter force will eventually, when the particle has passed, determine the state of motion of the electrons, but it will be negligible during the actual encounter and, therefore, of no influence on the energy transfer, provided only

$$\mu\omega_a^2\eta \ll \mu\ddot{\eta} \tag{3.7}$$

for $|t| \lesssim \frac{p}{\gamma v}$, or, according to (3.1) and (3.3), if

$$p \ll d_a = \frac{v}{\omega_a}\gamma. \tag{3.8}$$

The limiting distance d_a just corresponds to a collision time comparable with the proper period of the oscillators and it is, indeed, evident that, in case of shorter impulses, the energy balance is independent of the binding forces. For collisions of larger duration, however, these forces will essentially reduce the energy transfer. In fact, in the extreme case of $p \gg d_a$, the electron will with high approximation pass through a succession of equilibrium states and, finally, be left in its original position.

A more detailed calculation of the energy transfer to an electron bound in a quasi-elastic field of force leads (N. Bohr 1913, 1915) to the following expression for the stopping effect in distant collisions

$$S_q = B\left(\log\frac{k\gamma v}{q\omega_a} - \frac{1}{2}\frac{v^2}{c^2}\right), \tag{3.9}$$

where k is a numerical factor equal to 1.123. Formula (3.9) is seen to coincide with (3.5) for a value of $p_{\max}$ closely equal to d_a given by (3.8). It may be noted that, in the deduction of (3.9), it is assumed that q may be chosen small compared with d_a. For the treatment of the atomic interaction problems it is required (cf. § 2) that $q \gg a_0$, and the two conditions are thus compatible only for $d_a \gg a_0$. The problem of larger frequencies ω_a for which $d_a \gtrsim a_0$ may, however, be neglected in the present connection, since in that case the atomic binding forces produce a screening already at distances where the interaction effects are negligible.

§ 4. Atomic Interaction for Non-Relativistic Particle Velocities.

Turning now to the problem of the mutual interaction of the atoms in the stopping material, we shall see that this phenomenon may be characterized essentially as a further screening effect. In fact, when the electrons during the passage of the particle are displaced from their equilibrium positions, the medium is polarized and, hence, each atomic electron will be subjected to a restitutional force from the surrounding material.

It will be convenient first to confine ourselves to the more simple case of non-relativistic particle velocities, where the problem can be treated quite analogously to the influence of the internal atomic binding forces discussed in § 3. Introducing the field vectors $\boldsymbol{E}$ and $\boldsymbol{D}$, we note that, in the quasi-electrostatic approximation corresponding to $v \ll c$, we have rot $\boldsymbol{E} = 0$ and, therefore, also rot $\boldsymbol{D} = 0$, assuming the medium to be homogeneous and isotropic. This last relation will hold irrespective of the dispersion properties of the substance. Thus, $\boldsymbol{D}$ is determined from the same equations as, and must equal, the field surrounding the particle in vacuo. Now, the average electric field in the medium is given by $\boldsymbol{E} = \boldsymbol{D} - 4\pi \boldsymbol{P}$, where $\boldsymbol{P}$ is the dipole moment per unit volume and, consequently, the polarization produces a force on the electrons, equal to $4\pi e \boldsymbol{P}$.

Since $\boldsymbol{P}$ is given by $-ne\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is the electronic displacement vector and n is the density of electrons, the polarization

force is seen to be of the quasi-elastic type, corresponding to a cyclic frequency ν given by

$$\nu^2 = \frac{4\pi n e^2}{\mu} \tag{4.1}$$

and representing the frequency with which "free" electrons may oscillate in the medium. The influence of the atomic interaction forces on the motion of the electrons may, thus, be treated exactly like the effect of the atomic binding forces and will, in particular, imply a screening at a distance d_ν given by

$$d_\nu = \frac{v}{\nu} \tag{4.2}$$

in analogy to (3.8) for $\gamma = 1$.

It should be noted that the total force with which the medium acts on an electron may differ from $4\pi e \boldsymbol{P}$, corresponding to the well-known fact that the actual average field $\boldsymbol{F}$ to which the electrons are subjected will, in general, deviate from $\boldsymbol{E}$. In simple dielectrica like gases, where the neutral molecules may be regarded as independent entities, it may, thus, be shown that $\boldsymbol{F}$ equals $\boldsymbol{E} + \frac{4\pi}{3}\boldsymbol{P}$, and, also in denser materials, the same relation between $\boldsymbol{F}$ and $\boldsymbol{E}$ will hold in certain cases. Still, it is of particular interest for the following discussion to note that, if the electrons are not bound to certain fixed positions, but move all over space, as in metals or ionized materials, $\boldsymbol{F}$ and $\boldsymbol{E}$ will coincide (cf. DARWIN 1934). More generally, we may put $\boldsymbol{F} = \boldsymbol{E} + 4\pi a \boldsymbol{P}$, where a is a numerical constant characteristic of the structure of the substance. Since, however, the additional force $\boldsymbol{F} - \boldsymbol{E}$ may be ascribed to the effect of the atoms in the immediate neighbourhood of the electron considered, it will be convenient to include it in the atomic binding force $-\mu\omega_a^2\boldsymbol{\xi}$. If, thus, ω_a' represents the binding frequency of an isolated atom, we have,

$$\omega_a^2 = \omega_a'^2 - a\nu^2, \tag{4.3}$$

where ν is given by (4.1). As is well known from dispersion theory, ω_a will then represent the absorption frequency of the substance.

The combined influence of the atomic binding and the polarization may be treated by introducing an effective frequency ω_A defined by

$$\omega_A^2 = \omega_a^2 + \nu^2 \tag{4.4}$$

to replace ω_a in formulae deduced for isolated atoms. In particular, the stopping power may be obtained in this manner from the non-relativistic approximation of (3.9).

In order to estimate the significance of the atomic interaction effects in stopping problems, we must thus compare ν with the atomic frequency ω_a. From (4.1) we have, introducing the electronic velocity v_0 and the radius a_0 of the hydrogen atom,

$$\nu^2 = 4\pi n a_0^3 \left(\frac{v_0}{a_0}\right)^2. \tag{4.5}$$

Now, $\frac{v_0}{a_0}$ represents the order of magnitude of the frequencies of the most loosely bound electrons in atoms and since, even in dense materials, $n a_0^3$ is always smaller than unity, it follows that, compared with atomic frequencies, ν will never be very large and, in most cases, actually quite small. It is just for this reason that, in the non-relativistic problem which we have hitherto considered, the atomic interaction effects are usually of only secondary importance for the stopping power. Still, as already mentioned in the Introduction, there are exceptions to this rule. In fact, for the free electrons in metals or ionized substances, the binding frequency vanishes and, in this case, which we shall discuss more closely in § 5, the polarization effects become of decisive importance.

In the present paragraph, we shall further show how the stopping power of a substance may be directly described as a force with which the medium acts on the moving particle[1]. This alternative way of approach also allows of a simple deduction of the stopping formula which corresponds to (3.9) when due account is taken of the polarization effects. For illustration we shall, however, first briefly consider the analogous problem for the two-body collision.

1) This method has been outlined in a more qualitative way by N. Bohr (1948, § 3.1).

Instead of calculating the energy loss of the particle from the momentum transfer to the electron, one might in fact directly have estimated the reactive force of the struck electron. Thus, in the non-relativistic case considered here, the displacement of the electron implies that, taking the particle to be positive, the decelerative force in the last half of the collision more than compensates the acceleration in the first half. That part of the reactive force, directed against the motion of the particle, which is produced by the electronic displacement is given by

$$\delta K = -\eta \frac{\partial}{\partial p} \frac{z_1 e^2 v t}{(p^2 + v^2 t^2)^{3/2}} = \frac{3 z_1 e^2 \eta p v t}{(p^2 + v^2 t^2)^{5/2}} \tag{4.6}$$

and, by introducing η from (3.3), putting $\gamma = 1$, one finds for the resulting decrease in kinetic energy of the particle

$$\int_{-\infty}^{\infty} \delta K \, v \, dt = 3 \frac{z_1^2 e^4}{\mu v^2} v^3 \int_{-\infty}^{\infty} \frac{t^2 \, dt}{(p^2 + v^2 t^2)^{5/2}} = T \tag{4.7}$$

as given by (3.4).

In case of a particle penetrating through a substance, it is in a similar way the polarization which acts as a brake on the particle. If the medium is homogeneous and isotropic, however, no free charges will be generated (div $\boldsymbol{P} = 0$) except at the position of the particle. Inside the medium, therefore, each volume element remains neutral and gives rise to no resultant force, but along the path of the particle opposite charge will be accumulated. Of course, such considerations depending on average quantities like free charge cannot be applied to the material in the immediate neighbourhood of the particle but, for the purpose of considering the interaction between the particle and the medium at distances large compared with atomic dimensions, we may imagine removed a cylindrical tube of radius $q \gg a_0$ around the path of the particle. The force S_q with which the more distant part of the medium acts on the particle may then be calculated from the attraction of the free charges induced on the inner surface of this cylindrical tube.

The surface density σ of these charges equals $-en\eta$. For the value of η, however, we may not use the simple formula (3.3),

since we must take into account the presence of a harmonic force of frequency ω_A. If, however, we choose $q \ll \frac{v}{\omega_A}$, an electron at the surface will, during the time when the direct force of the particle is active, behave as if it were free and, assuming it to be at rest before the encounter, we get from (3.3)

$$\sigma(x) - \sigma(-x) = 2en \frac{z_1 e^2}{\mu v^2} \frac{x}{q}, \tag{4.8}$$

where $x = -vt$ denotes the distance of the electron from the instantaneous position of the particle, measured in the direction of v. The expression (4.8) will hold for $|x| \ll \frac{v}{\omega_A}$ but, for larger values of $|x|$, the harmonic force becomes of importance. Since, however, at such large distances the direct influence of the particle on the electronic motion perpendicular to the path is negligible, (4.8) may be simply generalized to

$$\sigma(x) - \sigma(-x) = 2en \frac{z_1 e^2}{\mu v^2} \frac{v}{\omega_A q} \sin\left(\frac{x\omega_A}{v}\right). \tag{4.9}$$

For the force acting on the particle, we thus have

$$S_q = z_1 e \int_0^\infty \frac{\sigma(x) - \sigma(-x)}{(x^2 + q^2)^{3/2}} 2\pi q x \, dx = B \frac{v}{\omega_A} \int_0^\infty \frac{x \sin\left(\frac{x\omega_A}{v}\right)}{(x^2 + q^2)^{3/2}} dx, \tag{4.10}$$

where B is given by (3.6). This integral can be expressed in terms of a Hankel function and gives asymptotically for $q \ll \frac{v}{\omega_A}$

$$S_q = B \log \frac{kv}{\omega_A q}, \tag{4.11}$$

corresponding to (3.9) for $v \ll c$, if only ω_a is replaced by ω_A.

The results of this paragraph, expressed by the formulae (4.1), (4.4), and (4.11), correspond for non-relativistic velocities to those obtained by the more general treatment of FERMI (1940), who, in order to cover the case of $v \sim c$, proceeds by a formally rather different method in which S_q is estimated as the flux of the Poynting vector through the surface of the cylinder of radius q.

§ 5. Stopping Power of Materials Containing Free Electrons.

As already mentioned, the atomic interaction effects are, for non-relativistic particle velocities, of special importance if the substance contains free electrons. Of particular interest in this respect is the stopping power of metals, where the conduction electrons may to a large extent be regarded as free (cf. § 10a).

A few remarks would seem required to justify an application of the considerations in the previous paragraph to problems of free electrons. Indeed, we have here in a sense to do with infinitely large atoms and the very definition of distant collisions, as encounters with impact parameter large compared with atomic dimensions, is therefore, strictly speaking, ambiguous. Still, to our purpose, it is not essential that the electrons are able to move freely throughout space, but we may imagine them confined within limited volumes of linear dimensions a, if only the corresponding oscillation frequency, which will be of the order of $\frac{\hbar}{\mu a^2}$, is small compared with ω_A. This condition may be fulfilled and a at the same time chosen small in comparison with the screening distance $\frac{v}{\omega_A}$, provided $\mu v^2 \gg \hbar \omega_A$. For smaller particle velocities, the stopping mechanism here considered is of only minor significance (cf. the concluding passage in § 2).

In the estimate of the stopping power, it must be taken into account that also other effects than the polarization will tend to restrain the electrons from moving freely. In fact, during the encounter with the incident particle, the electron may collide with ions or electrons in the medium. The influence of these collisions may be compared with the effect of a frictional force $-\mu \omega_\varrho \dot{\xi}$, where $\dot{\xi}$ is the velocity vector and where $\frac{1}{\omega_\varrho}$ is a measure of the time interval in which the electronic momentum is substantially changed. In particular for metals, it is well known from the theory of conduction that

$$\omega_\varrho = \frac{ne^2}{\mu} \varrho, \tag{5.1}$$

where ϱ is the specific resistance.

In an early treatment of the stopping power of metals, by v. WEIZSÄCKER (1933), it was actually suggested that the limit of effective interaction between the particles and the free electrons was determined by the resistance. As pointed out by KRAMERS (1947), however, such effects will, in general, be of only very small influence as compared with that of the polarization of the medium. In fact, since the momentum transfer from the particle to a free electron is comparable with the force of the particle multiplied by the effective collision time $\frac{p}{v}$, it follows that a frictional force can influence the collisions only for $p \gtrsim d_\varrho = \frac{v}{\omega_\varrho}$. Now, in metals at ordinary temperatures, $\omega_\varrho \ll \nu$ and, thus, $d_\varrho \gg d_\nu$ given by (4.2). Consequently, the effective adiabatic limit is primarily determined by the polarization.

A closer estimate of the influence of the friction on the stopping power may be obtained in complete analogy to the considerations leading to (4.11), the only difference being that (4.9) must now represent a damped oscillation. Thus, we merely have to add an extra factor $\exp\left\{-\frac{\omega_\varrho}{2v}x\right\}$ and substitute for ω_A the effective oscillation frequency which, for $\omega_a = 0$, will be equal to $\sqrt{\nu^2 - \frac{1}{4}\omega_\varrho^2}$. Evaluating S_q, one thereby finds asymptotically, for q small compared with the adiabatic limit,

$$S_q = B\left\{\log\frac{kv}{q\nu} - \frac{\omega_\varrho}{\sqrt{4\nu^2-\omega_\varrho^2}}\operatorname{arctg}\frac{\sqrt{4\nu^2-\omega_\varrho^2}}{\omega_\varrho}\right\}. \qquad (5.2)$$

This formula is equivalent to that obtained by KRAMERS by a somewhat different method and also coincides with the non-relativistic approximation of FERMI's formula[1]. For $\omega_\varrho \ll \nu$, the last term in the brackets in (5.2) has the approximate value $-\frac{\pi}{4}\frac{\omega_\varrho}{\nu}$, and the stopping formula therefore reduces to (4.11) for $\omega_A = \nu + \frac{\pi}{4}\omega_\varrho$. In the opposite extreme case of $\omega_\varrho \gg \nu$, the last

[1] In the case of bound electrons, there may likewise be a damping to take into account, e. g., due to radiative forces. In this more general case we get, of course, an expression for S_q which follows from (5.2) by simply replacing ν with ω_A given by (4.4) and ω_ϱ with the damping constant (cf. FERMI 1940).

term in the brackets in (5.2) has the asymptotic value $\log \frac{v}{\omega_Q}$ so that (5.2) now coincides with (4.11) for $\omega_A = \omega_Q$, just corresponding to the result of v. WEIZSÄCKER.

It may be noted that, in the above calculations, we have assumed the electrons to be at rest before the collision with the particle, whereas in metals or ionized materials the electrons actually have quite considerable velocities, often greatly surpassing the velocity changes induced by the particle. However, this circumstance should have no essential effect on the average energy transfer. In fact, if we denote by $\boldsymbol{w}$ the initial velocity, and by $\boldsymbol{u}$ the velocity alteration, the increase in kinetic energy is given by

$$T = \frac{1}{2}\mu(\boldsymbol{w}+\boldsymbol{u})^2 - \frac{1}{2}\mu w^2 = \frac{1}{2}\mu u^2 + \mu \boldsymbol{w}\boldsymbol{u} \qquad (5.3)$$

and, averaged over all directions of $\boldsymbol{w}$, the last term vanishes.

Still, of course, it must be assumed that the electrons, during the actual collisions, do not move over distances comparable with the impact parameter and that, therefore, w must be small compared with the particle velocity. In most cases of importance, this condition is amply fulfilled but, e. g. in ionized media at very high temperatures, the thermal velocities may exceed v even for "fast" particles and, under such circumstances, the whole stopping phenomenon acquires an essentially different character. However, we shall not here enter more closely on this problem.

§ 6. Atomic Interaction for Relativistic Particle Velocities.

The preceding considerations regarding the polarization effects were confined to particle velocities small compared with that of light. While, in this case, it was seen that, with the few exceptions discussed in § 5, the atomic interaction is of only minor importance, being in general of small influence compared with that of the internal binding forces, the situation is essentially different for relativistic velocities. In fact, for sufficiently large values of $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$, the radius of action of the particle is,

as shown by FERMI (1940), always determined by the polarization effects.

As regards collisions with single atoms, the modifications to be taken into account for $v \gtrsim c$ are of very simple character. Thus, as long as we are within the adiabatic limit, the energy transfer in distant collisions is, according to (3.4), independent of relativity effects. In fact, the retardation merely implies a contraction by a factor γ of the field of the incident particle in the direction of motion and an intensification of the field in the same ratio, and, therefore, does not affect the total momentum transfer. On account of the contraction of the field and the resulting shortening of the collision time, however, the adiabatic limit is increased by a factor γ, as also follows from (3.8).

It may be added that the influence of resistive forces discussed in § 5 is modified in a similar manner. Since a frictional force $-\mu\omega_\varrho\dot{\xi}$ is comparable with the force from the particle only if the collision time is of the order of or larger than $\frac{1}{\omega_\varrho}$, the screening distance corresponding to such effects will be given by

$$d_\varrho = \frac{v}{\omega_\varrho}\gamma, \tag{6.1}$$

which, for velocities small compared with that of light, reduces to the estimate in § 5.

An analysis of the atomic interaction effects in the relativistic case presents, however, a somewhat more intricate problem. In particular, we may no longer, like for $v \ll c$, compare the polarization force with a simple harmonic restitutional force. In fact, in contrast to the screening effect of a force of this type, one finds that the adiabatic limit for a material in which the electrons are free, and where the polarization is determining for the stopping effect, is uninfluenced by retardation effects. This result, which follows from FERMI's formula and which, as we shall see, can also be obtained by more elementary considerations, shows that in the relativistic case the mutual influence of the electrons is much stronger than corresponding to the electrostatic forces considered in § 4. Indeed, as was to be expected, all interaction effects become, for $\gamma \gg 1$, primarily of electromagnetic character.

In order to illustrate this latter point, it is instructive to consider a simple two-body collision from the point of view of the reaction of the struck electron on the incident particle. This problem was treated in § 4 for $v \ll c$, but is essentially modified by retardation effects. In fact, the particle will not "know" that the electron has been perturbed before the pulse caused by this perturbance catches up with the particle and, for velocities very close to c, this will happen a comparatively long time after the actual collision is initiated. Thus, a signal emitted by the electron at $t = 0$ will reach the particle at a distance $p\gamma$. Now, the periods characterizing the electronic motion will be comparable with the collision time $\frac{p}{v}\frac{1}{\gamma}$ and, for $v \lesssim c$, the field which this motion produces will, thus, mainly contain harmonic components of wave-lengths of the order of $\frac{p}{\gamma}$. Since such wave-lengths are small compared with the distances in question, it follows that the reaction of the electron on the particle is primarily determined by the electromagnetic wave field emitted by the accelerating electron. The electrostatic part of the field which depends on the electronic displacement and which, in the non-relativistic case, is responsible for the reaction, is here of only secondary importance.

The following estimate may serve to illustrate that the acceleration of the electron, for $\gamma \gg 1$, is actually determining for the reaction. In fact, the electric field intensity produced by an accelerated electron is, at large distances r, asymptotically given by

$$E(t) = \frac{e\ddot{\eta}(t')}{c^2 r} \sin\varphi, \tag{6.2}$$

where φ is the angle between the acceleration and the radius vector and where, as indicated, $\ddot{\eta}$ is to be taken at the retarded time $t' = t - \frac{r}{c}$. Since this field is transverse, i. e. perpendicular to r, the corresponding force acting on the particle against the direction of motion will be

$$\delta K(t) = \frac{z_1 e^2}{c^2} \frac{\ddot{\eta}(t')}{r} \sin\varphi \cos\varphi. \tag{6.3}$$

Now, during the actual collision, $\ddot{\eta}$ is, according to (3.1), comparable with $\frac{z_1 e^2 \gamma}{p^2}$ and since, as already mentioned, the r-values in question are of the order of $p\gamma$, we may, for $\gamma \gg 1$, put $\sin \varphi = 1$ and $\cos \varphi = \frac{1}{\gamma}$. Furthermore, the pulse given out by the accelerated electron, although it has a spatial extension of about $\frac{p}{\gamma}$, will act on the particle through a distance comparable with $p\gamma$, since the velocities of pulse and particle only differ by a relative amount of the order of γ^{-2}. It will thus be seen that the force component (6.3) gives rise to an energy loss just of the order of T given by (3.4).

A more accurate analysis of the reaction in the two-body collision is given in Appendix I, but the above cursory considerations suffice to illustrate the decisive part played by the radiation field. It is also just this circumstance which is manifested in the peculiar radiation effects which accompany the passage of very fast particles through matter and which will be discussed more closely in the next paragraph.

On similar lines as the simple analysis of the two-body collision, one may obtain an estimate of the mutual interaction

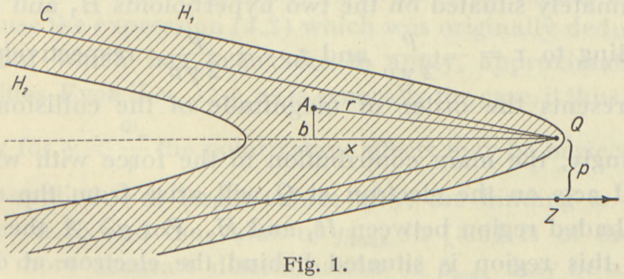

Fig. 1.

between the electrons in the penetrated substance. To this purpose, consider an electron at point Q (see Fig. 1), which is colliding with the particle Z passing at distance p. At the same time, the electron is acted upon by the surrounding electrons, and the major contribution will come from those electrons which, at the retarded time $t' = t - \frac{r}{c}$, were themselves accelerated, i. e. were colliding with the particle. Now, an electron at point A will, at the retarded

time, be in a phase of collision which, as compared with that of the electron at Q at the instant considered, is earlier by a time interval τ given by

$$\tau = \frac{r}{c} - \frac{x}{v}, \tag{6.4}$$

where x is the projected distance from Q to A measured against the direction of motion of the particle. Introducing $r^2 = x^2 + b^2$ (see the figure), one gets from (6.4)

$$\frac{(x + v\tau\gamma^2)^2}{v^2\tau^2\gamma^2(\gamma^2 - 1)} - \frac{b^2}{c^2\tau^2(\gamma^2 - 1)} = 1, \tag{6.5}$$

which shows that the points of constant τ are situated on a hyperboloid. In particular, the electrons which at the retarded time were in the same stage of collision as the electron at Q, will be found on the conical surface C extending backwards from Q and having an opening angle ϑ for which $\sin\frac{\vartheta}{2} = \frac{1}{\gamma}$. This surface intersects the path of the particle at the distance $x = \frac{v}{c}\gamma p$.

In the figure, the electron at Q is, at the instant considered, at the peak of collision. The electrons which, at the retarded time, were "beginning" or "concluding" their collisions are, thus, approximately situated on the two hyperboloids H_1 and H_2, corresponding to $\tau = -\frac{p}{2\gamma v}$ and $\tau = +\frac{p}{2\gamma v}$, respectively, since $\frac{p}{\gamma v}$ represents the order of magnitude of the collision time[1]. Accordingly, the main contribution to the force with which the material acts on the electron at Q will arise from the electrons in the shaded region between H_1 and H_2. For $\gamma \gg 1$, the essential part of this region is situated behind the electron at distances of the order of or smaller than $p\gamma$. In fact, further away, we have contributions from electrons accelerated in all directions, and therefore the total field of these electrons is small.

According to (6.2), the field from each electron is inversely proportional to r; furthermore, since $\sin\varphi \sim 1$ and since the majority

[1] Strictly speaking, it should be taken into account that the collision time varies with the impact parameter. For the present purpose of estimating orders of magnitude, however, the more cursory considerations should suffice, since the major part of the polarization force arises from atoms with impact parameters comparable with that of the atom at Q.

of the electrons in question have accelerations comparable with that of the electron at Q, the resulting polarization force K will be of the order of magnitude of

$$K \sim \frac{e^2}{c^2} \ddot{\eta} p^2 n \tag{6.6}$$

and directed against $\ddot{\eta}$. This force is evidently small compared with the total force $\mu \ddot{\eta}$ acting on the electron, only if

$$p \ll \frac{c}{\nu}, \tag{6.7}$$

where ν is given by (4.1). This upper limit should, therefore, represent the screening distance d_ν corresponding to the atomic interaction effects. The estimate of the polarization force is, of course, of a rather cursory character but, due to the complicated calculations which would be implied, we shall not attempt a detailed analysis from the microscopic point of view. The above considerations suffice, however, to bring out the essential point that, for $v \to c$, the distance d_ν approaches a constant value of the order of c/ν.

In estimating the stopping power of a substance we now have to compare d_ν with the adiabatic limit d_a corresponding to the effect of the binding and given by (3.8). As to the former distance, we may use the expression (4.2) which was originally deduced for $v \ll c$, but which has been seen to apply, approximately, for all velocities. Even if $\omega_a \gg \nu$, as is generally the case, it thus follows that only for $\gamma < \frac{\omega_a}{\nu}$ the polarization effects may be neglected and (3.9) be applied. For larger values of γ, the binding is of little influence on S, and the value of $p_{\max}$ in (3.5) is of the order of d_ν. A more accurate determination of $p_{\max}$ may be obtained by noting that the stopping power of a substance containing free electrons (cf. p. 27) is not affected by retardation. For large γ, we thus have the asymptotic expression.

$$S_q = B \log \frac{kv}{p\nu} \tag{6.8}$$

corresponding to (4.11) for $\omega_A = \nu$.

We have here neglected the possible influence of frictional forces which may become significant if d_ϱ given by (6.1) is

smaller than both d_a and d_ν. Under such circumstances, the stopping formula is obtained from (3.9) by substituting ω_ϱ for ω_a.

These results are in complete agreement with the more rigorously deduced formulae given by FERMI (1940). Only in the transitional cases, in which neither of the three distances d_a, d_ν or d_ϱ is very much smaller than the two others, certain refinements have to be introduced in analogy to the more detailed expressions given in § 4 and § 5 for non-relativistic velocities. Such corrections, however, amount at most to only a few per cent of the stopping power.

§ 7. Čerenkov Effect and its Relation to the Stopping Problem.

As already mentioned in the Introduction, the passage of very fast particles through dense matter is accompanied by a peculiar radiation (Čerenkov effect). An analysis of this phenomenon was first given by FRANK and TAMM (1937), who pointed out its immediate connection with the circumstance that the phase velocity of light in the substance may be smaller than the speed of the particle. In fact, the Čerenkov radiation presents a close analogue to familiar acoustical and hydrodynamical phenomena produced by an object moving with a velocity exceeding that of the wave velocities in the medium (TAMM 1939).

Since the phase velocity of electromagnetic waves is given by $c/\sqrt{\varepsilon}$, where ε is the dielectric constant, the radiation will take place over spectral regions for which

$$\sqrt{\varepsilon} > \frac{c}{v}. \tag{7.1}$$

Moreover, since the waves must be stationary with respect to the moving particle, the angle of emission for a frequency for which (7.1) is fulfilled is given by

$$\cos \Theta = \frac{c}{v\sqrt{\varepsilon}}, \tag{7.2}$$

where Θ is measured from the direction of motion of the particle.

In the present paragraph, we shall discuss, by means of simple arguments, some of the general characteristics of the Čerenkov effect and, in particular, its relation to stopping theory.

In order to obtain a survey of the various aspects of the phenomenon, it is of interest to consider the problem from the microscopic as well as from the macroscopic point of view. In the former case, we have immediate connections to the considerations of § 6, while the latter approach is more in analogy to that of FRANK and TAMM, and of FERMI.

From the microscopic point of view, the Čerenkov effect simply originates in the circumstance that part of the energy transferred from the penetrating particle to the electrons in the substance may be subsequently emitted as coherent radiation. Thus, it was not necessary in the previous paragraph to take the effect explicitly into account since, in problems of stopping power, one need, in the first instance, consider only the behaviour of the struck electrons during the actual collision with the particle. In fact, the energy loss of the particle may be said to be decided within this short time interval and is not affected by the question of the later distribution of the energy transferred to the electrons. In particular it is, from such considerations, immediately evident that the Čerenkov effect corresponds to part of the stopping power estimated in § 6 and should not be regarded as an additional source of energy loss (cf. FERMI 1940).

Some of the main features of the radiation may also be understood from an analysis like that in § 6. Thus, an emission of coherent radiation will demand the fulfilment of proper phase relations between the wavelets originating from the individual electrons, and this condition leads immediately to (7.2). Moreover, the spectral distribution is correlated to the rate at which the energy of the electronic oscillations is dissipated into radiation. For an isolated atom, this rate is very low, but it may be strongly increased by the influence of surrounding atoms. In fact, just due to the phase relations, the superposition of the electromagnetic fields of the individual oscillators may lead to greatly enhanced radiative effects.

In order to estimate the influence of atomic interaction on the Čerenkov spectrum, we may consider the two extreme cases, $d_\nu \gg d_a$ and $d_\nu \ll d_a$, in which, according to the considerations in § 6, the interaction forces may be regarded as, respectively, very weak and very strong compared with the binding forces. In the former case, we should expect the atoms to perform a large number of oscillations before their energy is radiated, and the

emitted spectrum to consist of a narrow line around the proper frequency of oscillation. In the latter case, however, the atoms will not be able to perform even a single oscillation and the frequency distribution should bear no simple resemblance to a spectral line.

Such general features of the radiation are just in accordance with those implied by the condition (7.1) for the spectrum. In fact, the dispersion law corresponding to the simple atomic model considered, involving only a single proper frequency ω_a, may be written

$$\varepsilon = 1 + \frac{\nu^2}{\omega_a^2 - \omega^2}, \tag{7.3}$$

where ν is given by (4.1). Furthermore, according to (3.8) and (4.2), the two cases in question correspond to $\nu\gamma \ll \omega_a$ and $\nu\gamma \gg \omega_a$, respectively. It is, therefore, seen that, in the former case, (7.1) is fulfilled only in a narrow region around ω_a while, in the latter case, it holds for all frequencies smaller than ω_a.

From the macroscopic point of view, the energy loss of the particle appears to take place in two essentially different modes. In fact, neglecting absorption due to damping forces, energy may either be radiated or it may be absorbed by the matter, giving rise to oscillations persisting in the medium after the passage of the particle. For the distinction between these two mechanisms, it is convenient to divide the electromagnetic field produced by the particle in the substance into a transverse (divergence-free) and a longitudinal (irrotational) component. The radiative part of the field is obviously of the transverse character, while the residual oscillations, left in the "wake" of the particle, must correspond to a longitudinal field. In fact, in the absence of electric currents, a divergence-free field consists of free electromagnetic radiation which propagates to infinite distances.

The longitudinal component just represents the field calculated with neglect of retardation effects and is, therefore, simply that considered in § 4, where the velocity of light was regarded as infinite. It is thus immediately seen that the energy absorbed by the medium in the neighbourhood of the path of the particle is given by the non-relativistic stopping formula[1].

1) It must be noted that, when damping is taken into account, also some part of the transverse field energy may be absorbed by the medium close to the particle (cf. § 8).

The relativistic increase of the stopping effect is due to the transverse field and, hence, represents the radiated energy. Moreover, it is evident from such considerations that, while the energy stored in the medium corresponds to that transferred to the atoms within the non-relativistic adiabatic limit, the radiation is emitted by the atoms at larger distances from the path of the particle.

In particular, it is of interest to note that, for substances containing free electrons ($\omega_a = 0$), it follows from (7.3) that ε is always smaller than unity, and the condition (7.1) can, thus, never be fulfilled. Consequently, no radiation occurs and the stopping power must, therefore, as already mentioned in § 6, for all velocities be given simply by the expression originally deduced for $v \ll c$.

For the residual field left in the medium after the passage of the particle, we have, of course, $\boldsymbol{D} = 0$ since, from the macroscopic point of view, no "true" charges are present. The dielectric constant of the medium must, therefore, vanish for the oscillation frequency concerned and, according to (7.3), this condition will just be fulfilled for $\omega = \omega_A$ given by (4.4), which was seen in § 4 to represent the proper frequency of the substance. It may be added that, from the very circumstance that we have to do with the excitation of oscillators of proper frequency corresponding to $\varepsilon = 0$, it may immediately be concluded that their energy absorption is unaffected by retardation effects. In fact, this energy depends, as is well known, exclusively on the resonance component of the exciting field, for which the phase velocity of light, $c/\sqrt{\varepsilon}$, is infinite.

In the evaluation of the radiated energy, Frank and Tamm (1937) and Fermi (1940) expand the electromagnetic field produced by the particle in harmonic components with respect to time-dependence. It is, however, also possible to adapt to the case of ponderable media the well-known method of radiation theory in which the field is dissolved in plane waves. In Appendix II, we shall consider the application of this formalism to the problems of stopping power and Čerenkov radiation. The method sheds some light on the phenomena in question and, in addition, is illustrative of the difference in approach between ordinary procedures of classical electromagnetic theory and the formalism which has become the conventional tool in quantum electrodynamics.

§ 8. The Atom as a General Dispersive System.

The preceding considerations have been based on a highly simplified atomic model in which the virtual oscillators were considered to have all the same frequency. This simplification was made in order to bring out as clearly as possible the principal points regarding the atomic interaction effects, but in a more detailed treatment, the atom should be compared with an ensemble of oscillators corresponding to the different excitation possibilities (cf. § 2). The proper frequencies of these oscillators will be denoted by ω_i, and their relative strengths by f_i normalized per electron $\left(\sum_i f_i = 1\right)$. If the atoms are bound together, this model is still adequate if only the oscillators represent the transition possibilities of the electrons in the molecules or in the lattice.

Such refinements are readily accounted for in the usual stopping theory. In fact, if the oscillators can be regarded as independent, formula (3.9) is simply to be replaced by

$$S_q = B \sum_i f_i \left(\log \frac{kv\gamma}{q\,\omega_i} - \frac{1}{2}\frac{v^2}{c^2} \right), \tag{8.1}$$

where the electron density n entering in the expression (3.6) for B is equal to $z_2 N$, if z_2 denotes the atomic number of the substance and N the number of atoms per unit volume.

The polarization phenomena, however, introduce a coupling between the different oscillators. In principle, this effect presents no great difficulties, since the calculation of FERMI (1940), or a procedure like that used in Appendix II, may be immediately generalized by replacing the simplified dispersion law (7.3) by

$$\varepsilon = 1 + \sum_i \frac{f_i\, \nu^2}{\omega_i^2 - \omega^2}, \tag{8.2}$$

corresponding to the atomic model on which (8.1) is based. Still, exact calculations by means of such methods lead to rather complicated expressions[1] and it is, therefore, of interest that, on the

1) The case of two dispersion frequencies has been considered by HALPERN and HALL (1940). The more general model corresponding to (8.2) has been treated by STERNHEIMER (1946), but an evaluation of the expressions deduced is difficult and has been attempted only under simplifying assumptions. The accuracy involved seems, therefore, hardly to go beyond that of the more simple analysis given here, the results of which also coincide in essentials with those obtained by STERNHEIMER.

See also Postscript (i), p. 50.

basis of considerations analogous to those in § 6, it is possible to obtain simple formulae representing a degree of accuracy sufficient for most purposes.

In such a microscopic treatment, the influence of the polarization on the stopping effect may be estimated by comparing the distance d_v with the adiabatic limits d_i, corresponding to the various atomic frequencies ω_i and given by (3.8) for $\omega_a = \omega_i$. The value of d_v will be given by an expression of the type of (4.2), but an essential point will be to estimate the effective electronic density determining for the polarization. In fact, due to the influence of the binding forces, this density will decrease with increasing distance from the path of the particle.

At distances comparable with d_i, the number of electrons per unit volume which contribute materially to the polarization will be equal to nF_i, where F_i represents the sum of the oscillator strengths, corresponding to atomic frequencies equal to or smaller than ω_i. According to the estimates in § 6, the atomic interaction will therefore be effective, provided $\gamma v F_i^{1/2} > \omega_i$, where v is given by (4.1) for $n = z_2 N$. For a survey of the problem, it will thus be convenient to introduce a critical frequency ω_c defined by

$$\omega_c = \gamma v F_c^{1/2}, \tag{8.3}$$

where

$$F_c = \sum_{\omega_i \leqq \omega_c} f_i. \tag{8.4}$$

In general, we may assume that equation (8.3) has only a single root, a point to which we shall return briefly in § 9, where approximate expressions for the frequency distribution of the oscillators are considered.

In this case, the situation is especially simple, and it is seen that the limit of free energy transfer will be determined primarily by the binding forces or the polarization, according as $\omega_i > \omega_c$ or $\omega_i < \omega_c$, respectively. The contribution to the stopping power of the former oscillators will approximately be given by the respective terms in (8.1), while for the latter oscillators the expression (6.8) for an effective electron density nF_c will apply.

The total stopping power of distant collisions may, thus, be written

$$S_q = B\left\{F_c \log\frac{kv}{q\nu F_c^{1/2}} + \sum_{\omega_i > \omega_c} f_i\left(\log\frac{kv\gamma}{q\omega_i} - \frac{1}{2}\frac{v^2}{c^2}\right)\right\}. \qquad (8.5)$$

In particular, it may be noted that, for very large velocities for which γv exceeds the largest significant atomic frequencies, we have $F_c \approx 1$ and, in this case, the stopping formula will be especially simple, being practically independent of the atomic frequencies. The latitude involved in (8.5) arises mainly from the estimate of the contribution of the oscillators with $\omega_i \sim \omega_c$, for which, of course, the polarization as well as the binding forces have a significant influence. The accuracy of the above approximation would, however, seem to be quite high since, as already mentioned in § 6, even in the case where all frequencies are equal and, therefore, may all fall in the transition region, the necessary corrections will never exceed a few per cent[1].

The part of the energy loss which is radiated to large distances may, according to the considerations in § 7, be readily estimated, provided the absorption due to damping effects can be disregarded. In this case, the radiated energy, in fact, simply represents the difference between (8.5) and the stopping power which would be obtained by disregarding relativity effects. Actually, however, we have to do with a considerable absorption in the spectral region extending from the lowest proper frequency of the substance to the highest relevant atomic frequencies, and an estimate shows that, in this region, by far the greater part of the radiation will, in not too dilute materials, be reabsorbed close to the path of the particle. It would thus seem that, to a first approximation, an actual emission of radiation, easily accessible to observation, will be confined to the region below the first absorption band. For such frequencies, which in generally comprise primarily the visible and infrared region and possibly part of the ultraviolet, the radiation spectrum may be calculated from formula (12) in Appendix II by introducing the proper values for the dielectric constant.

Finally, we shall consider briefly the influence of atomic interaction on the number of ions produced by the particle along the path. This problem involves in principle a detailed investigation of the distribution of the energy loss on the various atomic

[1]) Cf. Postscript (ii), p. 50.

oscillators, but it appears that the relationship between stopping power and ionization is not essentially affected by the polarization effects, and recourse may therefore be taken to the results of ordinary penetration theory. In the analysis of the problem, we may conveniently make use of the considerations in § 6 and § 7 and shall, in particular, divide the field surrounding the particle into a longitudinal and a transverse part.

The former part, corresponding to the energy loss calculated with neglect of retardation, will (cf. § 4) be only little influenced by the polarization, especially in case of dilute materials like gases, where the ionization problem is of particular importance. For this part of the interaction between particle and matter, we may thus immediately use the result, derived for collisions with isolated atoms, that in distant encounters the contribution of any bound electron to the number of primary ionization processes is closely proportional to the corresponding contribution to the stopping power. In the simplest case, of hydrogen, it follows in particular from the detailed calculations of BETHE (1930) that the number of ions produced per unit path, in collisions with impact parameter greater than q, is given by

$$P_q = S_q \frac{0.285}{\hbar \omega_0}. \tag{8.6}$$

For heavier substances, generalized approximate expressions may be given (cf. N. BOHR 1948, § 3.4).

The relativistic increase in the energy loss was seen to be correlated to the transverse part of the field, but it is of importance that, according to the above considerations, only a negligible part of the energy transfer due to the interaction with the ionization oscillators will be emitted as radiation to larger distances, since this energy is mainly concentrated in frequency regions of strong absorption.

A detailed investigation of the energy absorbed by the various oscillators from the transverse field is rendered difficult by the circumstance that the radiation emitted by one type of oscillators may, as discussed in § 7, contain frequencies extending over a wide interval and may, consequently, be absorbed by oscillators of a different type. However, just in case of gases, where v is very small compared with atomic frequencies, this "mixing" effect

should be rather insignificant. In fact, even in case of $d_i \gg d_v$, where the frequency distribution of the emitted radiation is most strongly spread out, it may be shown that, for $v \ll \omega_i$, the larger part of the energy is concentrated on frequencies which differ from ω_i by an amount small in comparison with ω_i itself. The mixing will, therefore, primarily take place between very close lying levels and should not essentially affect the relative number of ionization processes.

Relations of the type of (8.6) must thus, in general, be expected to be only little influenced by the atomic interaction effects. As regards the total ionization, including primary as well as secondary processes, one may likewise conclude that, as has been deduced for collisions with isolated atoms (cf., e. g., FANO 1946), the average energy expenditure per ion is largely independent of the particle velocity.

§ 9. Estimate of Stopping Power for Heavy Substances.

It follows from the considerations in the previous paragraph that the influence of atomic interaction may imply a considerable simplification in the stopping formula since, for very large values of $v\gamma$ corresponding to $F_c \approx 1$, the stopping becomes independent of the atomic frequencies and is determined only by the electronic density of the substance. In order, however, to evaluate S_q in the transition region where the polarization gradually becomes effective, it is necessary to investigate the distribution of the atomic oscillator frequencies involved in expressions like (8.5). In the case of heavy substances, a detailed analysis of this problem is complicated[1], but more cursory estimates, sufficient for many purposes, may be derived on the basis of simplifying assumptions regarding the frequency distribution.

In penetration problems, one thus often obtains an approximate account of general features (cf. N. BOHR 1948, § 3.5) by representing the sum F of the oscillator strengths corresponding to $\omega_i \leq \omega$ by an expression of the simple type

$$F = \left(\frac{\omega}{z_2^s \omega_0}\right)^{1/2}, \tag{9.1}$$

[1] Cf. Postscript (iii), p. 51.

where ω_0 denotes the Rydberg frequency. Of course, such a relation needs modification for the most loosely as well as the most firmly bound electrons in the atom. While the corrections in the low frequency region are of only minor significance in the present connection, it will be necessary to make adjustments in

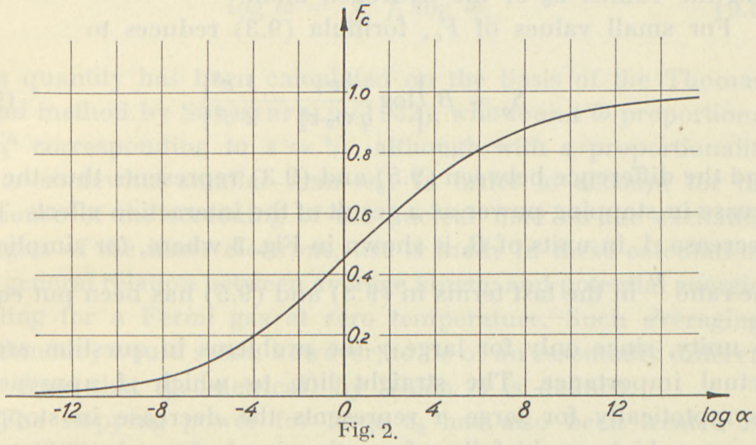

Fig. 2.

the high frequency region so as to take into account that the total oscillator strength equals unity. To this purpose, it would seem natural to put tentatively

$$F = \frac{\left(\frac{\omega}{z_2^s \omega_0}\right)^{1/2}}{1 + \left(\frac{\omega}{z_2^s \omega_0}\right)^{1/2}} \tag{9.2}$$

as a simple function which corresponds to (9.1) for $F \ll 1$ and gives $F = 1$ for large ω.

In particular, it may be noted that, for distributions of the type (9.1) or (9.2), the equation (8.3) determining for the interaction effects will have only a single root. The stopping power will, thus, be given by (8.5) and one finds, by replacing the sum by an integral and introducing (9.2),

$$S_q = B \left\{ \log \frac{k v F_c^{3/2}}{q \nu} - \frac{1}{2} (1 - F_c) \frac{v^2}{c^2} \right\} \tag{9.3}$$

as a simple approximate formula involving only F_c. The variation of this quantity in the transition region is shown in Fig. 2 which gives F_c as a function of α defined by

$$a = \left(\frac{\gamma \nu}{z_2^s \omega_0}\right)^2 = 16 \pi \gamma^2 N a_0^3 z_2^{1-2s}, \tag{9.4}$$

where the ratio between ν, given by (4.1), and the Rydberg frequency ω_0 has, for convenience, been written in terms of N and the radius a_0 of the hydrogen atom.

For small values of F_c, formula (9.3) reduces to

$$S_q = B \left\{ \log \frac{k v \gamma}{q \omega_0 z_2^s} - \frac{1}{2} \frac{v^2}{c^2} \right\} \tag{9.5}$$

and the difference between (9.5) and (9.3) represents thus the decrease in stopping power as a result of the interaction effects. This decrease Δ, in units of B, is shown in Fig. 3 where, for simplicity, the ratio $\frac{v}{c}$ in the last terms in (9.3) and (9.5) has been put equal to unity, since only for large γ the problems in question are of actual importance. The straight line to which Δ approaches asymptotically for large a represents the decrease in stopping power, which would follow from the simple formula (6.8) to be applied when the polarization effects have reached full efficiency.

In order to estimate the exponent s in (9.2) which gives the best fit to the actual frequency distribution of the atomic oscillators, we may compare (9.5) with theoretical and experimental deter-

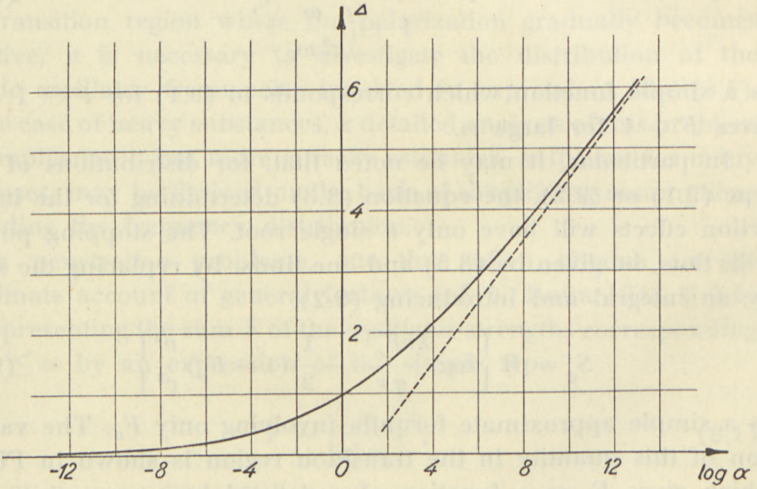

Fig. 3.

minations of the stopping power for heavy atoms in cases where the polarization effects are negligible. The expression (9.5) corresponds to (8.1) for $\overline{\omega} = \omega_0 z_2^s$, where $\overline{\omega}$ represents the average excitation frequency defined by

$$\log \overline{\omega} = \sum_i f_i \log \omega_i. \tag{9.6}$$

This quantity has been calculated on the basis of the Thomas-Fermi method by SOMMERFELD (1932), who found $\overline{\omega}$ proportional to $z_2^{4/3}$ corresponding to $s = 4/3$, although with a proportionality factor somewhat smaller than ω_0. In order to account for the influence of the screening of the nuclear field on the excitation energies of the inner electrons, use is made in these calculations of a general relation between average kinetic and potential energies holding for a Fermi gas at zero temperature. Such averagings (arithmetic) would seem, however, to be of an essentially different type from that (geometrical) by which $\overline{\omega}$ is defined.

The stopping power for large z_2 has also been treated by BLOCH (1933b) who, likewise, compared the atom with a Fermi gas, but considered explicitly its dynamic properties. Although the details of the distribution of the proper frequencies involved highly complicated calculations, it was found that F depends only on the ratio ωz_2^{-1}, a result which leads to $\overline{\omega}$ proportional to z_2. Later, the problem has been reconsidered by H. JENSEN (1937), who pointed out minor corrections to the results of BLOCH, but these refinements are of little importance in the present connection.

The estimate of BLOCH is confirmed by experiments of WILSON (1941) on the stopping power of protons which, in the region where the polarization effects are insignificant, is found to be in good agreement with the theory, if $\overline{\omega}$ is taken proportional to z_2. Moreover, the proportionality factor is estimated to be very nearly equal to ω_0 and the results, therefore, just correspond to (9.5) for $s = 1$. This circumstance may perhaps be taken as an indication of the approximate adequacy of the procedure used in the present paragraph to estimate the polarization effects in the transition region.

§ 10. Comparison with Experimental Evidence.

a) Non-relativistic velocities.

Considering, first, the case of particle velocities small compared with that of light, it was seen in § 4 and § 5 that the atomic interaction effects are of particular importance when the substance penetrated contains free electrons. Of special interest in this respect is the problem of metals.

According to the dispersion theory of metals (cf., e. g., KRONIG 1929, 1931), the behaviour of the valence electrons may be represented by a series of oscillators the first of which has zero frequency, corresponding to free electrons, and the remainder of which represent the transitions of the electrons between different bands (Brillouin zones). The most prominent of these transitions will have frequencies of the order of

$$\omega_b = \frac{\pi^2 \hbar}{2 \mu b^2} \tag{10.1}$$

if, for simplicity, we compare the lattice with a cubic structure of spacing $b = N^{-1/3}$. Denoting the number of valence electrons per atom by $z_2 F_v$, we have thus, by means of (4.1),

$$\frac{F_v v^2}{\omega_b^2} = \frac{16}{\pi^3} \frac{b}{a_0} z_2 F_v \tag{10.2}$$

and since, in general, b will be equal to 2 or 3 Å units, while $a_0 = \frac{1}{2}$ Å, it follows that, even for monovalent metals, the interaction between the valence electrons will overshadow the binding in the lattice. Since, furthermore, for the electrons in the interior atomic shells, the polarization is negligible, formula (8.5) should represent a fair approximation if F_c is simply put equal to F_v[1].

For the effective average excitation potential $I = \hbar \bar{\omega}$, we thus have (cf. (9.6))

$$I = F_v \log \hbar \nu F_v^{1/2} + \sum_i{}' f_i \log \hbar \omega_i, \tag{10.3}$$

where the summation is to be extended to the transition possibilities of the electrons bound in the atomic shells lying below the conduction band.

[1]) Cf., however, Postscript (ii), p. 50.

The most favourable circumstances for testing this formula are found in the lightest metals like lithium and beryllium, where the relative number of conduction electrons is largest. In Li, we have $F_v = \frac{1}{3}$ and $\hbar v F_v^{1/3} = 7.6\ eV$ while, for Be, one obtains $F_v = \frac{1}{2}$ and $\hbar v F_v^{1/3} = 18\ eV$. For comparison, it may be mentioned that $\hbar \omega_b$ is about 4.5 eV and 9 eV in Li and Be, respectively, and the corrections to the effective frequency of the valence electrons due to the lattice binding may, thus, from expressions like (4.4), be estimated to be only of the order of 5 %. It is, however, of interest to note that, in the gaseous state, where the polarization is negligible, we must reckon with effective excitation energies for the valence electrons of the order of the ionization potentials, which are 5.4 eV for Li and 9.3 eV for Be. The atomic stopping power of the metals must, therefore, be expected to be appreciably smaller than for the corresponding gases.

An evaluation of the sum in (10.3) involves, of course, a detailed analysis of the binding of the inner electrons, but just in case of Li and Be the problem is comparatively simple, since the terms represent only the excitations of the K-electrons. An estimate of the average excitation potentials for the K-shell has been made by LIVINGSTON and BETHE (1937, p. 264), who give 110 eV and 205 eV for Li and Be, respectively. By means of (10.3) these values lead to $I_{Li} = 45\ eV$ and $I_{Be} = 60\ eV$. It may be noted that we here assume the total oscillator strength $\sum' f_i$ for the K-electrons to be equal to $\frac{2}{z_2}$. This value should actually be somewhat decreased since the presence of outer electrons may prevent certain transitions from the K-shell (LIVINGSTON and BETHE, 1937). For Li and Be, however, the effect would appear negligible; in fact, not only is the number of L-electrons very small compared with the cases considered by LIVINGSTON and BETHE, but the effect even vanishes in the approximation in which the binding of the outer electrons can be represented by s-states.

The stopping power of Li has been determined by ROSENBLUM (1928), whose results indicate a value of I of about 40 eV (cf. MANO 1933). However, the experimental uncertainty of about 10 % in the stopping power corresponds to a latitude in

I_{Li} of 50 %. For Be, no reliable measurements appear available, but empirical evidence would be of considerable interest due to the high density of valence electrons which should imply a comparatively small stopping power[1].

It may be added that the influence of the metallic resistance is in general quite negligible (cf. KRAMERS 1947). Thus, the value of ω_ϱ given by (5.1) corresponds, in the case of Li at normal temperatures, to $\omega_\varrho = 0.01\, v F_v^{1/2}$. The use of (5.2) for the valence electrons may, thus, be estimated to increase the above value for I_{Li} by less than 1 %. On the other hand, v. WEIZSÄCKER's theory in which ω_ϱ is assumed to determine the adiabatic limit for the oscillator of zero frequency leads to $I_{Li} = 10\ eV$, which is decidedly at variance with the empirical data. Furthermore, this theory predicts a considerable temperature dependence of the stopping power which, for a decrease in resistance by a factor 100, should increase by about 20 % in the case of fast α-rays. Experiments by GERRITSEN (1946) have shown that no such temperature variation occurs.

Also in other substances than those actually containing free electrons, the polarization effect may be of some significance for the stopping already for $v \ll c$. In fact, in solid or liquid materials, the values of $\hbar v F^{1/2}$ will, even for $F z_2 \sim 1$, most frequently be of the order of 10 eV and may, therefore, exceed the excitation energies of the most loosely bound electrons. In many cases, one may accordingly put $z_2 F_c$ in (8.5) equal to the number of bonding electrons, and it is of interest that the atomic interaction phenomena thus not only imply a certain reduction in stopping power, but also entail a simplification in the theoretical estimate of S. Indeed, under such circumstances, it is not necessary to consider details of the rather complicated mechanism of molecular binding or lattice structure, since the electrons involved will give practically the same contribution to S as if they were free.

b) Relativistic velocities.

As often mentioned in the preceding, the polarization phenomena become of special importance in the domain of relativistic

[1] Note added in proof. A recent investigation of the stopping power of protons in metallic Be (C. B. MADSEN and P. VENKATESWARLU, Phys. Rev. **74**. 648 (1948)) has given $I_{Be} = 64 \pm 5\ eV$ in good agreement with the above estimate.

velocities where, for sufficiently large values of γ, they imply a considerable reduction of the stopping power. In the original treatment of FERMI, attention was in particular called to the implications of this effect on the interpretation of measurements of fast cosmic ray particles.

A direct experimental test of the influence of atomic interaction on the stopping power is made difficult by the large energies required, but some evidence is given by the measurements of CRANE, OLESON and CHAO (1940) of the stopping in carbon of 10 *MeV* electrons. These investigators found an energy loss appreciably smaller than corresponding to formula (8.1), but (cf. HALPERN and HALL 1940) in good agreement with expression (6.8) which may be applied for the velocity in question. In fact, we have $F_c \approx 1$ since the value of $\hbar\nu\gamma$ is about 900 *eV* and, therefore, exceeds $z_2^2 \hbar\omega_0 \sim 500$ *eV*, representing the order of magnitude of the largest significant excitation energies of carbon.

Moreover, the influence of the polarization effects has been observed by ionization measurements. While it was shown by HAZEN (1945) in experiments with cosmic ray electrons that, for not too large values of γ, the ionization increases logarithmically with γ, corresponding to (8.1), some indication was obtained by HAYWARD (1947) that, for very large γ, the ionization reaches a constant value. Compared with the minimum in the ionization for $\gamma \sim 1$, the limiting value represents a relative increase which was found to conform, within the experimental latitude, with the theoretical estimate of the increase in stopping power, given by (6.8) and (8.1).

This evidence agrees with the considerations of § 8, according to which the atomic interaction, although of importance for the absolute values of ionization and stopping power, should have only minor influence on the ratio between these two quantities. In particular, it may be noted that this relationship rests on the assumption that the major part of the radiative energy loss correlated with the ionization oscillators is absorbed close to the path of the particle. Since the radiation represents the relativistic increase in the stopping effects, the observations of HAZEN, as well as those of HAYWARD, thus confirm the expected strong absorption.

As regards the interpretation of ionization measurements, it

may under circumstances be necessary to take into account that the average energy expenditure per ion, although largely independent of velocity, must still be expected to increase somewhat with γ due to the increasing importance of distant collisions relative to close collisions (cf. FANO 1946). As regards the measurements of HAYWARD, however, the increase in energy loss per ion from $\gamma \sim 1$ to very large γ may be estimated, from FANO's expressions, to be of the order of only 1 % in a He gas at normal pressure.

Finally, it may be recalled that the atomic interaction manifests itself very conspicuously in the radiative effects accompanying the penetration of fast particles, a phenomenon which has been investigated in detail since its discovery by ČERENKOV (1934). Thus, the general properties of the radiation, such as its polarization and the relation (7.2) between frequency and angle of emission, have been tested by ČERENKOV (1937, 1938) and by COLLINS and REILING (1938), and also the intensity of the radiation was shown by the latter investigators to be in accordance with theory. Recently, attempts have been made to use the radiation as a velocity indicator by applying the simple relation (7.2) for the directions in which the emission occurs (GETTING 1947, FURRY 1947, DICKE 1947).

Appendix I.

Reaction in Relativistic Two-body Collision.

In § 6, it was indicated in outline how the collision between two point charges may in classical mechanics be analyzed by tracing the reaction of the struck particle on the incident particle. We shall here consider this problem in some greater detail, confining ourselves, as in the text, to the treatment of distant collisions or, to be more specific, to the case of $p \gg \frac{z_1 e^2}{\mu v^2}$. In this approximation, the displacement, during the collision, of the struck particle, referred to as the electron, is small compared with p and its velocity remains negligible in comparison with c. Moreover, we may disregard the change of velocity of the incident particle.

The force which acts on the particle at time t depends on

the motion and position of the electron at the retarded time t' for which, on account of the simplifications mentioned, we have

$$t' = t - \frac{r}{c} = t - \frac{1}{c}\sqrt{p^2 + v^2 t^2}. \tag{I.1}$$

In the first approximation, the force is simply given by the static field corresponding to the electron at rest in its original position. Since, however, this field gives rise to no resultant energy transfer to the particle, we disregard it in the present connection. In higher approximations, we have reactive forces depending on the electronic displacement and motion. In the case of distant collisions, we need consider only linear terms in η, of which there will firstly be the force corresponding to a uniform motion of the electron. This force may simply be obtained from (4.6) by replacing $\eta(t)$ by $\eta(t') + (t - t')\,\dot{\eta}(t')$, representing the position which would have been reached by the electron at time t if it had proceeded from time t' with uniform velocity. Secondly, the acceleration of the electron at time t' produces a field given by (6.2).

For the total effective force of reaction one thus obtains, by means of (I.1),

$$\delta K(t) = \frac{z_1 e^2}{r^3} v t p \left\{ \frac{3\eta}{r^2} + \frac{3\dot{\eta}}{rc} + \frac{\ddot{\eta}}{c^2} \right\}, \tag{I.2}$$

where η and its derivatives are to be taken at the retarded time t'. This expression might, of course, also have been found from the general formula for the field produced by a point charge in arbitrary motion (cf., e. g., M. Abraham, Theorie der Elektrizität, 3rd ed., Leipzig and Berlin 1914, p. 92). The energy decrease of the particle is given by

$$T = \int_{-\infty}^{+\infty} \delta K\, v dt = T_1 + T_2 + T_3, \tag{I.3}$$

corresponding to the three terms in (I.2).

By a partial integration one finds

$$T_1 = z_1 e^2 p \int_{-\infty}^{\infty} \frac{1}{r^3} \frac{d\eta}{dt}\, dt = z_1 e^2 p \int_{-\infty}^{\infty} \frac{1}{r^3} \dot{\eta} \frac{dt'}{dt}\, dt, \tag{I.4}$$

where, according to (I.1),

$$\frac{dt'}{dt} = 1 - \frac{v}{c}\frac{vt}{r}. \tag{I.5}$$

Consequently, we have

$$T_1 + T_2 = z_1 e^2 p \int_{-\infty}^{\infty} \frac{\dot{\eta}}{r^3}\left(1 + 2\frac{v}{c}\frac{vt}{r}\right) dt \tag{I.6}$$

and, by another partial integration, again making use of (I.5),

$$T_1 + T_2 = \frac{z_1 e^2}{p}\left[\dot{\eta}\,\frac{t}{r}\right]_{t=-\infty}^{t=\infty} - T_3 - \frac{z_1 e^2}{p}\int_{-\infty}^{\infty} \ddot{\eta}\,\frac{t - \frac{r}{c}}{r}\,dt. \tag{I.7}$$

Now, from (I.1) it follows that $\frac{dt}{r} = \gamma\, dt'\,(\gamma^2 v^2 t'^2 + p^2)^{-1/2}$ and it is therefore seen, by a transformation of the last term in (I.7) to an integral over t', that this term vanishes, since $\ddot{\eta}$ is an even function of t' and since $t - \frac{r}{c} = t'$. Applying (3.2) for $\dot{\eta}$, one thus finds from (I.7) that $T = T_1 + T_2 + T_3$ coincides with the value given by (3.4).

An evaluation of T_1, T_2, and T_3, separately, involves somewhat more lengthy calculations which lead to rather complicated expressions, indicating that the division of T into three parts in the above manner is not of a very significant character. Putting $\beta = \frac{v}{c}$, one obtains

$$\begin{aligned} T_1 &= T\,(1-\beta) \\ T_2 &= T\left[-\frac{3}{\beta}(1-\beta^2) + \frac{3}{2}\frac{1-\beta^2}{\beta^2}\log\frac{1+\beta}{1-\beta}\right] \\ T_3 &= T\left[\frac{1}{\beta}(3-2\beta^2) - \frac{3}{2}\frac{1-\beta^2}{\beta^2}\log\frac{1+\beta}{1-\beta}\right]. \end{aligned} \tag{I.8}$$

In particular, it is seen that, as stressed in § 6, the term T_3, depending on the acceleration of the electron, becomes dominating for $\beta \to 1$.

So far, we have considered only the problem of the energy transfer but, for the sake of completeness, we shall briefly examine also how the momentum transfer in the relativistic two-body collision is described. The component perpendicular to the path of the incident particle presents no special problems; in fact, the momentum transferred in this direction to the electron was analyzed already in § 3, and the corresponding reactive force is, of course, simply the electrostatic force of the electron considered approximately at rest. It is of interest, however, that the latter circumstance is sufficient to show that, in distant collisions, the momentum transfer in this direction and, consequently, the total energy transfer is uninfluenced by relativity effects.

The problem of the component parallel to the path requires somewhat more detailed considerations. It is true that, in distant collisions, the momentum transfer in this direction is very small compared with that perpendicular to the path but, still, corresponding to the slowing down of the particle, there must, of course, be such a momentum transfer of magnitude $\frac{1}{v} T$. From the point of view of the reaction of the electron on the particle, this momentum is just that transferred by the force δK, given by (I.2).

In the non-relativistic case where actio equals reactio, the corresponding transfer of momentum from the particle to the electron is accounted for by the difference, due to the electronic displacement, in the electric force of the particle, in the first and last half of the collision. In the relativistic case, however, this force in the direction parallel to the path is given by

$$-eE_x = \frac{z_1 e^2 \gamma v t}{(p^2 + \gamma^2 v^2 t^2)^{3/2}} \tag{I.9}$$

and, by a simple calculation, one obtains from (3.3)

$$\int_{-\infty}^{\infty} e \frac{\partial E_x}{\partial p} \eta \, dt = \frac{1}{v} T (1 - \beta^2), \tag{I.10}$$

where T is given by (3.4). Thus, for $\beta \approx 1$, the electric field accounts for only a small part of the momentum transfer.

The remaining part is transferred by magnetic interaction. In fact, the magnetic field of the particle, which is directed perpendicular to the plane containing the electron and the path of the particle, equals βE_y, where E_y is the component of the electric field perpendicular to v. The transfer of momentum through magnetic forces is, thus, given by

$$\int_{-\infty}^{\infty} \frac{e\dot{\eta}}{c} \beta E_y \, dt = \frac{1}{v} T\beta^2 \tag{I.11}$$

since the integral of $e E_y \dot{\eta} \, dt$ just represents the energy transfer.

It should be stressed that the decisive part played by the radiative field of the struck electron in the slowing down of the particle, of course, in no way implies that the energy actually radiated during the encounter constitutes a major part of the energy transfer. In fact, this energy will be given by

$$W = \frac{2}{3} \frac{e^2}{c^3} \int_{-\infty}^{\infty} |\ddot{\boldsymbol{\xi}}|^2 \, dt, \tag{I.12}$$

neglecting the acceleration of the incident particle, the mass of which we may, for simplicity, consider to be very large. Introducing for $\ddot{\boldsymbol{\xi}}$ the force divided by μ, one obtains by a simple integration

$$W = \frac{2}{3} \frac{e^2}{c^3} \frac{z_1^2 e^4 \gamma^2}{\mu^2} \int_{-\infty}^{\infty} \frac{p^2 + v^2 t^2}{(p^2 + \gamma^2 v^2 t^2)^3} \, dt = T \frac{\pi}{8} \cdot \frac{e^2}{\mu c^2 p} \beta\gamma \left(1 + \frac{1}{3\gamma^2}\right). \tag{I.13}$$

Since, in distant collisions, p is extremely large compared with the classical electron radius $\frac{e^2}{\mu c^2}$, the value of W will thus be negligible compared with T. For excessively large values of γ, where the field of the particle at distance p is contracted to dimensions comparable with $\frac{e^2}{\mu c^2}$, the situation would be different, but such problems lie outside the scope of the simple classical picture of two colliding point charges.

Appendix II.

Application to Penetration Problems of Formalism of Radiation Theory.

As mentioned in § 7, it is possible to treat also ponderable media by the method, particularly well known from quantum electrodynamics (cf., e. g., W. HEITLER, Theory of Radiation I. 6, Oxford 1944), of dissolving the field in plane waves, and we shall here consider its application to the problems of stopping power and Čerenkov radiation. This method implies an expansion in Fourier components with respect to the spatial variation of the field and may, therefore, not always be well adapted to the case of dispersive media. Just for a field produced by a uniformly moving charge, however, the spatial components will also be harmonic in time.

We shall first treat the Čerenkov effect which is the more naturally suited to the formalism[1]. Since this phenomenon is connected with the rotational part of the field, we consider the transverse part of the vector potential which we expand in the familiar manner

$$\boldsymbol{A}_{tr} = \sum_{\lambda} q_\lambda \boldsymbol{A}_\lambda + q_\lambda^* \boldsymbol{A}_\lambda^* \quad \boldsymbol{A}_\lambda = \sqrt{4\pi c^2}\, \Omega^{-1/2} \boldsymbol{e}_\lambda \, e^{i(\boldsymbol{\varkappa}_\lambda \boldsymbol{r})}, \quad \text{(II.1)}$$

where q^* and $\boldsymbol{A}^*$ denote the complex conjugates of q and $\boldsymbol{A}$. The field is here assumed to be enclosed in a volume Ω, and the unit vector $\boldsymbol{e}_\lambda$ gives the direction of polarization. We follow the usual procedure in which terms corresponding to both $\boldsymbol{\varkappa}_\lambda$ and $-\boldsymbol{\varkappa}_\lambda \equiv \boldsymbol{\varkappa}_{-\lambda}$ are contained in the summation. The amplitudes q are then not uniquely defined by (II.1), but are determined by certain extra conditions imposed on their time dependence.

On account of the aforementioned difficulties in treating quite generally the case of dispersive media, we assume in the first instance the substance to have a constant value of ε. Neglecting specific magnetic properties of the medium, the field is given by

$$\Delta \boldsymbol{A} - \frac{\varepsilon}{c^2} \frac{\partial^2}{\partial t^2} \boldsymbol{A} = -\frac{4\pi \boldsymbol{i}}{c}, \quad \text{(II.2)}$$

where $\boldsymbol{i}$ denotes the current density corresponding to the moving particle. Multiplying this equation by $\boldsymbol{A}_\lambda^*$, and integrating over Ω, one thus finds

[1] Cf. Postscript (iv), p. 51.

$$(\ddot{q}_\lambda + \ddot{q}^*_{-\lambda}) + \omega_\lambda^2 (q_\lambda + q^*_{-\lambda}) = \frac{z_1 e}{\varepsilon c} (v, A^*_\lambda(x)), \qquad \text{(II.3)}$$

where x denotes the position of the particle, considered to be a point charge. The frequency ω_λ is given by

$$\omega_\lambda = \frac{\varkappa_\lambda c}{\sqrt{\varepsilon}}. \qquad \text{(II.4)}$$

We now require that the free waves associated with q_λ must have the time dependence $e^{-i\omega_\lambda t}$, which condition leads to

$$\ddot{q}_\lambda + \omega_\lambda^2 q_\lambda = \frac{z_1 e}{2\varepsilon c} \left(1 + \frac{i}{\omega_\lambda} \frac{d}{dt}\right) (v, A^*_\lambda(x)), \qquad \text{(II.5)}$$

an equation describing an oscillator in forced vibration.

If the particle moves with constant velocity, we may put $x = vt$ and the right hand side of (II.5) is harmonic with frequency $(\varkappa_\lambda, v)$. The equation (II.5) then reduces to

$$\ddot{q}_\lambda + \omega_\lambda^2 q_\lambda = \frac{z_1 e}{2 c \varepsilon \{(\varkappa_\lambda, v)\}} \left(1 + \frac{(\varkappa_\lambda, v)}{\omega_\lambda}\right) (v, A^*_\lambda(vt)). \qquad \text{(II.6)}$$

In this particular case, there is no difficulty in treating dispersive media; as indicated in (II.6), one simply inserts for ε the value corresponding to the frequency $(\varkappa_\lambda, v)$.

In vacuum, it follows from (II.4) that ω_λ is numerically greater than $(\varkappa_\lambda, v)$, since $v < c$. The solutions to (II.6) are, therefore, simple forced vibrations of constant amplitude. However, in a ponderable medium (or in the imaginary case of $v > c$) we may for certain wave numbers have $\omega_\lambda = (\varkappa_\lambda, v)$, corresponding to resonance between the exciting force and the oscillator. In this case, the oscillator will continue to absorb energy, corresponding to an actual emission of radiation. This effect just represents the Čerenkov radiation and it is also seen that, according to (II.4), the condition for resonance is identical with (7.2).

The treatment of an oscillator in resonance presents certain mathematical intricacies which may be overcome by introducing, formally, an infinitesimal damping. A more convenient representation is obtained, however, by making use of the Dirac δ-function. The general solution to (II.6) may thus be written, symbolically,

$$q_\lambda = \frac{z_1 e}{2\,c\,\omega_\lambda}\frac{1}{\varepsilon\{(\varkappa_\lambda, v)\}}\left(\frac{1}{\omega_\lambda - (\varkappa_\lambda, v)} + i\pi\delta\,(\omega_\lambda - (\varkappa_\lambda v))\right)(v, A_\lambda^*\,(vt))\,, \quad \text{(II.7)}$$

as may be easily verified[1].

In particular, we shall use the expression (II.7) to determine the force S_{tr} acting on the particle. For symmetry reasons, the Lorentz force obviously vanishes and we have, thus,

$$S_{tr} = -\frac{z_1 e}{c}\dot{A}_{tr}\,(vt) = -\frac{z_1 e}{c}\sum_\lambda \dot{q}_\lambda A_\lambda(vt) + \dot{q}_\lambda^* A_\lambda^*(vt) \quad \text{(II.8)}$$

which gives, by means of (II.7) and (II.1),

$$S_{tr} = -4\,\pi^2 z_1^2 e^2\,\Omega^{-1}\sum_\lambda e_\lambda\,(e_\lambda, v)\frac{(\varkappa_\lambda, v)}{\omega_\lambda\,\varepsilon\{(\varkappa_\lambda, v)\}}\,\delta(\omega_\lambda - (\varkappa_\lambda, v)). \quad \text{(II.9)}$$

Summing first over the two directions of polarization and introducing $(\varkappa_\lambda, v) = \varkappa_\lambda v y$, we get in the usual manner in the limit of infinitely large volumes Ω

$$S_{tr} = z_1^2 e^2 \int_0^\infty \varkappa^2\,d\varkappa \int_{-1}^{+1} dy\,\frac{\varkappa v^2 y}{\omega\,\varepsilon\{\varkappa v y\}}\,(1-y^2)\,\delta(\omega - \varkappa v y), \quad \text{(II.10)}$$

where S_{tr} is the component of S_{tr} directed against v. The other components vanish for symmetry reasons.

In evaluating the integral (II.10) it is convenient to change to the new variables ω, defined by (II.4), and $z = \frac{v}{c}\,y\sqrt{\varepsilon\{\varkappa v y\}}$. Since $v\,d\varkappa\,dy = d\omega\,dz$, one finds

$$S_{tr} = \frac{z_1^2 e^2}{c^2}\int_0^\infty \omega\,d\omega \int z\,dz\left(1 - \frac{c^2}{v^2\varepsilon\{\omega z\}}\right)\delta(1-z), \quad \text{(II.11)}$$

where the last integral is to be extended over values of z for

1) It may be shown that the equation $\ddot{x} + \omega_0^2 x = k(t)$ with the boundary condition $x = \dot{x} = 0$ for $t = -\infty$, in a Fourier expansion has the solution

$$x^\omega = k^\omega \frac{1}{2\,\omega_0}\left[\frac{1}{\omega_0 - \omega} + i\,\pi\,\delta\,(\omega_0 - \omega) + \frac{1}{\omega_0 + \omega} - i\,\pi\,\delta\,(\omega_0 + \omega)\right],$$

where $x(t) = \int_{-\infty}^{\infty} x^\omega\, e^{-i\,\omega\,t}\,d\omega$ and similarly for $k(t)$.

which $-1 < \frac{c\,z}{v\sqrt{\varepsilon\{\omega z\}}} < 1$. This integral vanishes, except for frequencies for which $z = 1$ is contained in the integration interval, i. e. for which $v\sqrt{\varepsilon\{\omega\}} > c$.

Thus, one gets finally

$$S_{tr} = \frac{z_1^2 e^2}{c^2} \int_{v\sqrt{\varepsilon}>c} \left(1 - \frac{c^2}{\varepsilon v^2}\right) \omega\, d\omega, \tag{II.12}$$

which is just the expression given by FRANK and TAMM (1937) for the Čerenkov radiation and its spectral distribution. For the simple dispersion formula (7.3) the expression (II.12) may easily be evaluated, and one finds the values given by FERMI (1940) which, as mentioned in § 7, represent the relativistic increase in the stopping formula.

The non-relativistic part of the stopping power is, as discussed in § 7, determined by the irrotational part of the field. It should be emphasized that the application to this problem of a formalism analogous to that used for the Čerenkov effect is somewhat artificial but, for the sake of completeness, we shall give a brief account of the procedure.

Choosing, for convenience, a gauge in which the vector potential is purely transverse, the longitudinal part of the field is given by the scalar potential φ, for which we have

$$\Delta\,\varphi = -\frac{4\pi\varrho}{\varepsilon}, \tag{II.13}$$

where ϱ denotes the charge density of the particle and where, like for the transverse field, we consider first the case of constant ε. Expanding φ, one gets, in analogy to (II.1),

$$\varphi = \sum_\sigma q_\sigma \Phi_\sigma + q_\sigma^* \Phi_\sigma^* \qquad \Phi_\sigma = \sqrt{4\pi c^2}\,\Omega^{-1/2}\, e^{i(\varkappa_\sigma, r)} \tag{II.14}$$

and may obtain, by considerations similar to those leading to (II.5),

$$\varepsilon\, q_\sigma = \frac{z_1 e}{2\,\varkappa_\sigma^2 c^2}\left(1 + \frac{i}{\omega_A}\frac{d}{dt}\right)\Phi_\sigma^*(x), \tag{II.15}$$

where ω_A represents the effective frequency defined by (4.4). As

discussed in § 7, only free oscillations of this particular frequency can be excited in the medium, and we have, accordingly, defined the variables q_σ so as to contain terms corresponding to free waves of the type $e^{-i\omega_A t}$, only. In case of a uniformly moving particle, we get from (II.15),

$$\varepsilon\{(\varkappa_\sigma v)\}\, q_\sigma = \frac{z_1 e}{2\,\varkappa_\sigma^2 c^2}\left(1+\frac{(\varkappa_\sigma, v)}{\omega_A}\right)\Phi_\sigma^*(vt) \qquad \text{(II.16)}$$

where, like in (II.6), we have taken into account the dispersion.

Now, the force acting on the particle is given by

$$S_{\text{long}} = -z_1 e\,\text{grad}\,\varphi = -z_1 e \sum_\sigma i\varkappa_\sigma (q_\sigma \Phi_\sigma(vt) - q_\sigma^* \Phi_\sigma^*(vt)) \qquad \text{(II.17)}$$

and, therefore, vanishes except for the contribution from the singularity in the terms representing wave-numbers for which $\varepsilon = 0$. It was to be expected, however, that only these components give rise to a stopping force, since the energy transfer to the medium takes place over the frequency ω_A for which just $\varepsilon = 0$ (cf. § 7).

In the neighbourhood of $\omega = \omega_A$ we may write the dispersion formula (7.3), by means of (4.4),

$$\varepsilon \approx -\frac{\omega_A^2 - \omega^2}{v^2} \qquad \text{(II.18)}$$

and, in complete analogy to the symbolism used in (II.7), we thus get

$$q_\sigma \approx -\frac{z_1 e}{2\,\omega_A}\frac{v^2}{\varkappa_\sigma^2 c^2}\left(\frac{1}{\omega_A - (\varkappa_\sigma, v)} + i\pi\delta(\omega_A - (\varkappa_\sigma, v))\right)\Phi_\sigma^*(vt). \qquad \text{(II.19)}$$

Introducing in (II.17) one finally gets, similarly to (II.10),

$$S_{\text{long}} = z_1^2 e^2 v^2 \int_0^\infty \varkappa\, d\varkappa \int_{-1}^{1} y\, dy \frac{1}{\omega_A}\delta(\omega_A - \varkappa v y) = \frac{z_1^2 e^2 v^2}{v^2}\int_{\frac{1}{v}\omega_A}^\infty \frac{d\varkappa}{\varkappa}. \qquad \text{(II.20)}$$

This expression, however, ceases to be valid for very large $\varkappa$,

since the whole procedure applies to distant collisions only. In analogy to § 4, where we considered the medium outside a certain distance q from the path of the particle, we may here introduce a cut-off at some wave number $\varkappa_{max}$. Choosing $\varkappa_{max} \sim \frac{1}{q}$ it is seen, by means of (3.6) and (4.1), that (II.20) coincides with (4.11) as closely as could be expected, considering the arbitrariness in cut-off procedure.

The present investigation has been carried out at the Institute for Theoretical Physics, Copenhagen, and the writer wishes to acknowledge the great benefit he has derived from the continual interest of Professor NIELS BOHR in the work. He is also grateful to Professor C. MØLLER for helpful discussions and to M. Sc. BØRGE MADSEN for preparing the figures.

Postscript Added in Proof.

Since the completion of the present manuscript, a number of investigations dealing with the same topics have come to the notice of the writer; some of these have been published only quite recently, while others were published already during the war, but the respective periodicals were not available in Copenhagen at the time in question.

(i) G. C. WICK (Ric. Scient. **11**, 273 (1940), ibid. **12**, 858 (1941), and Il Nuovo Cimento **9**, no. 3 (1943)) has considered the extension of FERMI's calculations to the multi-frequency model (cf. § 8) and, on the basis of a dispersion formula of the type (8.2), has worked out an expression for the reduction in stopping power, easily susceptible to numerical calculation for any given set of dispersion frequencies. The result had been obtained independently by O. HALPERN and H. HALL (Phys. Rev. **73**, 477 (1948)), the publication of whose work was delayed by the war. These latter authors moreover have deduced an explicit formula for the stopping power, valid under certain simplifying assumptions which are fulfilled in most cases of interest. This formula just coincides with (8.5); in fact, the approximation involved is equivalent to that underlying the analysis in § 8. It may be noted that the authors start from a dispersion law which, in contrast to (8.2), takes into account the Lorentz-Lorenz correction (cf. p. 12), but it would seem that, in the approximation considered, this correction may be neglected.

(ii) HALPERN and HALL (loc. cit.) consider also the influence of the damping of conduction electrons and point out that the effect, although in general negligible, may be of significance in special cases like that of carbon, where the resistance is excessively large. From the line of

approach adopted in § 8, this effect is readily taken into account; in fact, it follows from (6.1) that, as regards the stopping power, the conduction electrons are equivalent to dispersion oscillators of effective frequency ω_Q. Thus, the phenomenon may be said to be actually covered by (8.5).

(iii) The influence of the polarization on the stopping power has been computed for various substances by HALPERN and HALL (loc. cit.) on the basis of approximate dispersion-conduction models derived from X-ray ionization data. As pointed out by these authors, this procedure involves a certain latitude but, due to the relative insensitivity of the stopping effect on the exact model, the results may be expected to be reliable within a few per cent. More detailed estimates have been made, for a number of substances, by WICK (loc. cit.), who has employed X-ray data as well as theoretical calculations on the basis of the Hartree method in the establishment of appropriate sets of oscillators for the atoms in question.

(iv) V. I. GINSBURG (Journ. of Physics II, 441 (1940)) has treated the Čerenkov radiation by a Hamiltonian formalism which is very similar to the procedure applied in Appendix II. Such a formalism can be immediately quantized in the usual manner, and GINSBURG has developed a quantum electrodynamics which describes, in a phenomenological way, the radiation field in a ponderable medium. In particular, GINSBURG verifies that the average radiated energy is practically equal to that given by the classical formula, a result which was to be expected from quite general arguments (cf. § 2).

REFERENCES

H. A. Bethe, Ann. d. Phys. 5, **5**, 325 (1930).
F. Bloch, Ann. d. Phys. 5, **16**, 285 (1933a).
— ZS. f. Phys. **81**, 363 (1933b).
N. Bohr, Phil. Mag. **25**, 10 (1913).
— Phil. Mag. **30**, 581 (1915).
— Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. **18**, 8 (1948).
P. Čerenkov, C. R. Acad. Sci. URSS **2**, 451 (1934).
— Phys. Rev. **52**, 378 (1937).
— C. R. Acad. Sci. URSS **20**, 651 and **21**, 116, 339 (1938).
G. B. Collins and V. G. Reiling, Phys. Rev. **54**, 499 (1938).
D. R. Corson and R. B. Brode, Phys. Rev. **53**, 773 (1938).
H. R. Crane, N. L. Oleson and K. T. Chao, Phys. Rev. **57**, 555(A), (1940).
C. G. Darwin, Proc. Roy. Soc. **A 146**, 17 (1934).
R. H. Dicke, Phys. Rev. **71**, 737 (1947).
U. Fano, Phys. Rev. **70**, 44 (1946).
E. Fermi, Phys. Rev. **56**, 1242 (1939).
— Phys. Rev. **57**, 485 (1940).
I. Frank and Ig. Tamm, C. R. Acad. Sci. URSS **14**, 109 (1937).
W. H. Furry, Phys. Rev. **72**, 171 (A) (1947).
A. N. Gerritsen, Physica **12**, 311 (1946).
I. A. Getting, Phys. Rev. **71**, 123 (1947).
O. Halpern and H. Hall, Phys. Rev. **57**, 459 (1940).
E. Hayward, Phys. Rev. **72**, 937 (1947).
W. E. Hazen, Phys. Rev. **67**, 269 (1945).
H. Jensen, ZS. f. Phys. **106**, 620 (1937).
H. A. Kramers, Atti Congr. Int. di Fisica, Como (1927).
— Physica **13**, 401 (1947).
R. de L. Kronig, Proc. Roy. Soc. **A 124**, 409 (1929).
— Proc. Roy. Soc. **A 133**, 255 (1931).
M. G. Mano, C. R. **197**, 319 (1933).
N. F. Mott, Proc. Camb. Phil. Soc. **27**, 553 (1931).
A. Sommerfeld, ZS. f. Phys. **78**, 283 (1932).
M. Stanley Livingston and H. A. Bethe, Rev. Mod. Phys. **9**, 245 (1937).
R. M. Sternheimer, Diss., Univ. of Chicago, March 1946 (unpublished).
W. F. G. Swann, J. Franklin Inst. **226**, 598 (1938).
Ig. Tamm, Journ. of Phys. USSR **I**, 439 (1939).
C. F. v. Weizsäcker, Ann. d. Phys. 5, **17**, 869 (1933).
R. R. Wilson, Phys. Rev. **60**, 749 (1941).

Indleveret til Selskabet den 23. januar 1948.
Færdig fra trykkeriet den 15. januar 1949.